

The transformation properties of the gravitational energy-momentum in the teleparallel gravity are analyzed. It is proved that the gravitational energy-momentum in the teleparallel gravity can be expressed in terms of the Lorentz gauge potential, and therefore is not covariant under local Lorentz transformations. On the other hand, it can also be expressed in terms of the translation gauge field strength, and therefore is covariant under general coordinate transformations. A simplified Hamiltonian formulation of the teleparallel gravity is given. Its constraint algebra has the same structure as that of general relativity, which indicates the equivalence between the teleparallel gravity and general relativity in the Hamiltonian formulation.

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Gravitational energy-momentum and the Hamiltonian formulation of the teleparallel gravity

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I. INTRODUCTION

Recently, as a description of gravity equivalent to general relativity the teleparallel gravity has attracted renewed attention [1-5] owing to many salient features of it. First of all, the teleparallel gravity can be regarded as a translational gauge theory [1, 2, 4, 6], which make it possible to unify gravity with other kinds of interactions in the gauge theory framework.. In this direction interesting developments [7] have been achieved in the context of Ashtekar variables [8]. Another advantage of the teleparallel gravity concerns energy-momentum, its representation, positivity and localization [1, 2, 5]. In the context of general relativity no tensorial expression for the gravitational energy-momentum density can exist owing to the equivalence principle [9]. On the other hand, because of its simplicity and transparency the teleparallel gravity seems to be much more appropriate than general relativity to deal with the problem of the gravitational energy-momentum. It is proved that [10, 1, 2, 5] in the teleparallel gravity there exists a gravitational energy-momentum tensor which is covariant under general coordinate transformations and global Lorentz transformations. However, it is not covariant under local Lorentz transformations. The question arises if we can improve it further to obtain a energy-momentum object which is covariant under general coordinate transformations as well as local Lorentz transformations. An answer will be given in this paper.

The teleparallel gravity is characterized by a vanishing curvature and a nonvanishing torsion. The vanishing of the curvature may be a conceptual advantage of this formulation in the sense that it may actually define a background structure.. The identification and establishment of a background structure is an important issue to be considered since in quantum gravity, at least from the particle physics point of view, one would ultimately deal with the energy and momentum of the excitations of the gravitational field, and these excitations must defined with respect to a background structure.

For a teleparallel geometry there is a preferred class of frames, which greatly simplify computations. They can be obtained by selecting any frame at one point and parallel transporting it to all other points. Since the curvature vanishes, the parallel transport is path independent so the resultant frame field is globally well defined and then their transformations are also global, for example, a global Lorentz group. This means that the vanishing of the curvature or the existence of the absolute parallel (teleparallel) structure excludes the possibility of the localization of the global transformation group, e.g. Lorentz group and then prohibits introducing the corresponding gauge field. In such a teleparallel frame field the connection coefficients (Lorentz gauge fields) vanish. Teleparallel frame fields are unique up to a global (constant, rigid) linear transformation. In other words, an important feature of the teleparallel gravity is that the frame field transforms under a global Lorentz group. Consequently, the gravitational energy-momentum is a tensor with respect to coordinate transformations and a global Lorentz group but not a tensor with respect to a local Lorentz group. A proof of this conclusion will be given in Sec. II.

Attempts at identifying an energy-momentum density for gravity in the context of general relativity lead only to various energy-momentum complexes which are pseudotensors and then a new quasilocal approach which can be traced back to the early work of Penrose [11] has been proposed and become widely accepted [5, 12]. According to this approach quasilocal energy-momentum can be obtained from the Hamiltonian. Every energy-momentum pseudotensor is associated with a legitimated Hamiltonian boundary term. Hence, the pseudotensors are quasilocal and acceptable. In the Hamiltonian formulation of the teleparallel gravity a proof of the positive gravitational energy was obtained [5].

Concerning the problem of localization of gravitational energy-momentum, some Hamiltonian formulations of the teleparallel gravity with various gauge fixing have been presented recently [2, 3, 5]. Furthermore, a consistently established Hamiltonian formulation not only guarantees field variables to have a well defined time evolution but also

allow us to understand the physical meaning of the theory from a different perspective. The Hamiltonian formulation of general relativity reveals the intrinsic structure of the theory: the time evolution of the field is determined by the scalar and vector constraints. This is a essential feature for the canonical approach to the quantum theory of gravity. As is well known, the teleparallel gravity is equivalent to general relativity, it is naturally to ask if their Hamiltonian formulations have the same structures. It will be shown this is the case. In Sec. III a simplified Hamiltonian formulation of the teleparallel gravity is established and then in Sec. IV its constraint algebra is obtained under a new gauge fixing. One can find that not only the Hamiltonian but also the constraint algebra of the teleparallel gravity has the same structure as that of general relativity, which indicates the equivalence between teleparallel gravity and general relativity in the Hamiltonian formulation.

II. GRAVITATIONAL ENERGY-MOMENTUM AND ITS TRANSFORMATION PROPERTY

Usually, it is asserted on the basis of the principle of equivalence that the gravitational energy cannot be localized [9]. The principle of equivalence requires that the gravitational field can be made to vanish by a transformation in a sufficiently small region of the spacetime, which leads to recognizing the connection on the spacetime manifold as the strength of a gravitational field. However, if the gravitational energy-momentum density consists of the curvature rather than the connection like the case in electromagnetism, we would not have the problem of energy localization.

As is well known, a physical object has different transformation characters under different transformation groups. Therefore the answer to the question about the transformation characters of the gravitational energy-momentum object depends on the choice of variables, the choice of the gauge group and the expression of the energy-momentum object itself. For example, it depends on whether the expression of the energy-momentum object consists of the gauge potential or the gauge field strength. In a gauge theory the gauge potential is not covariant under the corresponding gauge group. As a result, the self current of the gauge field derived from Noether theorems is not covariant naturally [13].

In this section we will see that the gravitational energy-momentum can be expressed in terms of the Lorentz gauge potential, therefore it is not covariant under local Lorentz transformations. On the other hand it can also be expressed in terms of the translation gauge field strength and therefore is covariant under general coordinate transformations.

We start with a common relation between the tetrad e^I_μ , the spin connection ω_μ^{IJ} , and the affine connection $\Gamma^\rho_{\nu\mu}$ [14]

$$\partial_\mu e^I_\nu + \omega_\mu^{IJ} e^J_\nu - \Gamma^\rho_{\mu\nu} e^I_\rho = 0, \quad (1)$$

where $I, J, \dots = 0, 1, 2, 3$ are the internal indices and $\mu, \nu, \dots = 0, 1, 2, 3$ are the spacetime indices. If we define the Cartan connection [1,15]

$$\Gamma^\rho_{(c)\mu\nu} = e^\rho_\mu \partial_\nu e^I_\nu, \quad (2)$$

then (1) leads to

$$\begin{aligned} \Gamma^\rho_{(c)\mu\nu} &= \Gamma^\rho_{\mu\nu} - \omega_\mu^{IJ} e^I_\nu e^J_\rho \\ &= \Gamma^\rho_{\mu\nu} - \omega_\mu^{\rho\nu} \\ &= \{\omega_\mu^{\rho\nu}\} + K^\rho_{\mu\nu} - \omega_\mu^{\rho\nu}, \end{aligned} \quad (3)$$

where

$$\omega_\mu^{\rho\nu} = \omega_\mu^{IJ} e^I_\nu e^J_\rho, \quad (4)$$

and $\{\omega_\mu^{\rho\nu}\}$, $K^\rho_{\mu\nu}$ is the Christoffel connection and the affine contorsion, respectively. By introducing the Cartan torsion

$$T^\rho_{(c)\mu\nu} = \Gamma^\rho_{(c)\mu\nu} - \Gamma^\rho_{(c)\nu\mu}, \quad (5)$$

and the Cartan contorsion [1,15]

$$K^\rho_{(c)\mu\nu} = \frac{1}{2}(T^\rho_{(c)\nu\mu} + T_{(c)\mu}^\rho{}_\nu + T_{(c)\nu}^\rho{}_\mu), \quad (6)$$

we can obtain from (3)

$$T_{(c)\mu\nu}^\rho = T_{\mu\nu}^\rho - \omega_\mu^\rho{}_\nu + \omega_\nu^\rho{}_\mu, \quad (7)$$

and

$$K_{(c)\mu\nu}^\rho = K_{\mu\nu}^\rho + \omega_{\mu\nu}^\rho. \quad (8)$$

where

$$T_{\mu\nu}^\rho = 2\Gamma_{[\mu\nu]}^\rho = \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho, \quad (9)$$

is the affine torsion and

$$K_{\mu\nu}^\rho = \frac{1}{2}(T_{\mu\nu}^\rho + T_\mu^\rho{}_\nu + T_\nu^\rho{}_\mu), \quad (10)$$

is the affine contorsion [16].

In [14] the Cartan torsion $T_{(c)JK}^I = T_{(c)\mu\nu}^\rho e_\rho^I e_J^\mu e_K^\nu$ with a factor $\frac{1}{2}$ is called the anholonomy and denoted as C_{JK}^I , i.e.

$$C_{JK}^I = \frac{1}{2}T_{(c)JK}^I = \frac{1}{2}T_{(c)\mu\nu}^\rho e_\rho^I e_J^\mu e_K^\nu \quad (11)$$

and then is given a different geometric meaning. It is not a gauge-covariant object. In the 'holonomic gauge' C_{JK}^I vanishes and then we have a natural (or coordinate) coframe. In this case (7) gives the relation between the anholonomy, the affine torsion and the spin connection:

$$2C_{\mu\nu}^\rho = 2C_{JK}^I e_I^\rho e_\mu^J e_\nu^K = T_{\mu\nu}^\rho - \omega_\mu^\rho{}_\nu + \omega_\nu^\rho{}_\mu. \quad (12)$$

In the case of vanishing affine torsion

$$T_{\mu\nu}^\rho = 0, \quad (13)$$

as in the usually general relativity, we have

$$K_{\mu\nu}^\rho = 0, \quad (14)$$

and then

$$K_{(c)\mu\nu}^\rho = \omega_{\mu\nu}^\rho. \quad (15)$$

As a result (3) reads

$$\Gamma_{(c)\mu\nu}^\rho = \{\mu^\rho{}_\nu\} + K_{(c)\mu\nu}^\rho, \quad (16)$$

and we are led to the theories given by Hayashi, Shirafuji [15], de Andrade, Guillen and Pereira [1] which are equivalent to general relativity and called theories of teleparallel gravity. In these theories the curvature of the Cartan connection vanishes:

$$R_{(c)\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{(c)\sigma\nu}^\rho - \partial_\nu \Gamma_{(c)\sigma\mu}^\rho + \Gamma_{(c)\tau\mu}^\rho \Gamma_{(c)\sigma\nu}^\tau - \Gamma_{(c)\tau\nu}^\rho \Gamma_{(c)\sigma\mu}^\tau \equiv 0, \quad (17)$$

while the curvature of the Christoffel connection

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \{\sigma^\rho{}_\nu\} - \partial_\nu \{\sigma^\rho{}_\mu\} + \{\tau^\rho{}_\mu\} \{\sigma^\tau{}_\nu\} - \{\tau^\rho{}_\nu\} \{\sigma^\tau{}_\mu\} \quad (18)$$

does not. For these theories one can say that the spacetime is a Weitzenbock spacetime with respect to the Cartan connection or a Riemann spacetime with respect to the Christoffel connection.

Maluf develops another kind of teleparallel description of general relativity [2] in which the curvature of the affine connection vanishes while the affine torsion does not. Therefore one can say that the spacetime of the Maluf's description is a Weitzenbock spacetime with respect to the affine connection.

According to [1], the Lagrangian of the gravitational field can be chosen as

$$\mathcal{L}_G = -\frac{{}^{(4)}e c^4}{16\pi G} S^{\rho\mu\nu} T_{(c)\rho\mu\nu}, \quad (19)$$

where ${}^{(4)}e = \det(e^I_\mu)$, and

$$S^{\rho\mu\nu} = \frac{1}{2}(K^{\mu\nu\rho}_{(c)} - g^{\rho\nu} T^{\sigma\mu}_{(c)\sigma} + g^{\rho\mu} T^{\sigma\nu}_{(c)\sigma}). \quad (20)$$

The energy-momentum density of the gravitational field is:

$${}^{(4)}e j_I{}^\rho = -\frac{{}^{(4)}e c^4}{4\pi G} e_I{}^\lambda S_\mu{}^{\nu\rho} T_{(c)\nu\lambda}^\mu - e_I{}^\rho \mathcal{L}_G. \quad (21)$$

The quantity $j_I{}^\rho$ transforms covariantly under a general spacetime coordinate transformation, and is invariant under local translation of the tangent-space coordinates. However, it transform covariantly only under a global Lorentz transformation. How does it behave under a local Lorentz transformation? The answer is given in the following..

From (7) and (13) we can obtain

$$T_{(c)\mu\nu}^\rho = \omega_\nu{}^\rho{}_\mu - \omega_\mu{}^\rho{}_\nu, \quad (22)$$

and

$$S_\mu{}^{\nu\rho} = \frac{1}{2}(\omega_\mu{}^\nu{}_\sigma - \delta_\mu^\rho \omega_\sigma{}^{\sigma\nu} + \delta_\mu^\nu \omega_\sigma{}^{\sigma\rho}), \quad (23)$$

which lead to

$$S^{\rho\mu\nu} T_{(c)\rho\mu\nu} = \omega_\rho{}^{\mu\nu} \omega_\mu{}^\rho{}_\nu - \omega_\rho{}^{\rho\nu} \omega_\mu{}^\mu{}_\nu, \quad (24)$$

and

$$S_\mu{}^{\nu\rho} T_{(c)\nu\lambda}^\mu = \frac{1}{2} \omega_\mu{}^\rho{}_\nu (\omega_\lambda{}^\mu{}_\nu - \omega_\nu{}^\mu{}_\lambda) - \frac{1}{2} \omega_\mu{}^{\mu\nu} (\omega_\lambda{}^\rho{}_\nu - \omega_\nu{}^\rho{}_\lambda) - \frac{1}{2} \omega_\mu{}^{\mu\rho} \omega_\nu{}^\nu{}_\lambda. \quad (25)$$

The equations (21), (19), (24) and (25) indicate that $j_I{}^\rho$ consists of the Lorentz gauge potential $\omega_\rho{}^\mu{}_\nu$ algebraically and then is neither covariant nor invariant under local Lorentz transformations. If local Lorentz transformations are not introduced in the theory then $\omega_\rho{}^\mu{}_\nu = 0$ which leads to $j_I{}^\rho = 0$. Therefore, we are led to the conclusion that in teleparallel gravity the gravitational energy-momentum is not covariant under local Lorentz transformations because it consists of the Lorentz gauge potential.

Using (15), (6) and (5) we can compute

$$\begin{aligned} \omega_{\mu\nu\rho} &= K_{(c)\rho\mu\nu} = \frac{1}{2}(T_{(c)\mu\rho\nu} + T_{(c)\nu\rho\mu} + T_{(c)\rho\mu\nu}) \\ &= (e_{I\rho} \partial_{[\nu} e^I_{\mu]} + e_{I\mu} \partial_{[\rho} e^I_{\nu]} + e_{I\nu} \partial_{[\rho} e^I_{\mu]}), \end{aligned} \quad (26)$$

$$\omega^{\nu\mu\rho} = g^{\nu\tau} g^{\rho\sigma} e_I{}^\mu \partial_{[\sigma} e^I_{\tau]} + g^{\mu\lambda} g^{\rho\sigma} e_I{}^\nu \partial_{[\sigma} e^I_{\lambda]} + g^{\mu\lambda} g^{\nu\tau} e_I{}^\rho \partial_{[\tau} e^I_{\lambda]}, \quad (27)$$

$$\begin{aligned} S^{\rho\mu\nu} T_{(c)\rho\mu\nu} &= \omega_{\mu\nu\rho} \omega^{\nu\mu\rho} - \omega_\mu{}^{\mu\rho} \omega_\nu{}^\nu{}_\rho \\ &= \frac{1}{4} T_{(c)\rho\mu\nu} T_{(c)}^{\rho\mu\nu} + \frac{1}{2} T_{(c)\rho\mu\nu} T_{(c)}^{\nu\mu\rho} + T_{(c)\rho\mu}{}^\rho T_{(c)\nu}{}^{\nu\mu} \end{aligned} \quad (28)$$

Using the translation gauge field strength

$$F^I_{\mu\nu} = c^2 e_\rho^I T_{(c)\mu\nu}^\rho = c^2 (\partial_\mu e_\nu^I - \partial_\nu e_\mu^I), \quad (29)$$

the Eq. (19) can be written as

$$\mathcal{L}_G = \frac{{}^{(4)}e}{64\pi G} (\eta_{IJ} g^{\mu\lambda} g^{\nu\tau} + 2e_I{}^\tau e_J{}^\nu g^{\mu\lambda} - 4e_I{}^\mu e_J{}^\lambda g^{\nu\tau}) F^I_{\mu\nu} F^J_{\lambda\tau}. \quad (30)$$

The first term of (21) is

$$\begin{aligned} \frac{{}^{(4)}e c^4}{4\pi G} e_I{}^\lambda S_\mu{}^{\nu\rho} T_{(c)\nu\lambda}^\mu &= \frac{{}^{(4)}e}{8\pi G} e_I{}^\tau \left[\frac{1}{2} (e_J{}^\lambda e_K{}^\mu g^{\rho\nu} + e_J{}^\rho e_K{}^\mu g^{\lambda\nu} - \eta_{JK} g^{\lambda\mu} g^{\rho\nu}) \right. \\ &\quad \left. - e_J{}^\nu e_K{}^\rho g^{\lambda\mu} + e_J{}^\nu e_K{}^\lambda g^{\rho\mu} \right] F^J_{\mu\nu} F^K_{\lambda\tau}. \end{aligned} \quad (31)$$

From (20), (30) and (31) we can see that the current $j_I{}^\rho$ consists of the translation gauge field strength $F^I_{\mu\nu}$ and then is covariant under local translations. This makes teleparallel gravity different from the usual gauge field theories where the Noether current of the gauge fields is not covariant under corresponding gauge transformations.

III. 3+1 DECOMPOSITION

In order to obtain the Hamiltonian formulation of the theory a foliation in the spacetime manifold M should be introduced. Assuming that $M = \Sigma \times R$ for some space-like Manifold Σ , we can choose a time function t with nowhere vanishing gradient $(dt)_\mu$ such that each $t = \text{const}$ surface Σ_t is diffeomorphic to Σ . Introduce a time flow vector t^μ satisfying $t^\mu(dt)_\mu = 1$, we can decompose it perpendicular and parallel to Σ_t : $t^\mu = Nn^\mu + N^\mu$, where n^μ is the time-like normal at each point of Σ_t and N, N^μ are the lapse function and the shift vector, respectively. The spacetime metric $g_{\mu\nu}$ introduces a spatial metric $q_{\mu\nu}$ on each Σ_t by the formula

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (32)$$

a triad

$$e^I_i = q^\mu_i e^I_\mu \quad (33)$$

defined on Σ_t with $e = \det(e^I_i) = {}^{(4)}e/N$ and a volume element ϵ_{ijk} of q_{ij} . Since the affine torsion vanishes by (13), from this section we will drop the subscript (c) of the Cartan torsion $T^I_{(c)\mu\nu}$ and simply denote it by $T^I_{\mu\nu}$. Then the Lagrangian

$$\mathcal{L}_G = \frac{{}^{(4)}ec^4}{64\pi G} (\eta_{IJ} g^{\mu\lambda} g^{\nu\tau} + 2{}^{(4)}e_I{}^\tau {}^{(4)}e_J{}^\nu g^{\mu\lambda} - 4{}^{(4)}e_I{}^\mu {}^{(4)}e_J{}^\lambda g^{\nu\tau}) T^I_{\mu\nu} T^J_{\lambda\tau} \quad (34)$$

can be rewritten as

$$\begin{aligned} \mathcal{L}_G = \frac{Nec^4}{64\pi G} & (T_{ijk} T^{ijk} + 2T_{ikj} T^{jki} - 4T^i_{ki} T^{jk}_j - T^\perp_{ij} T^{\perp ij} + 4T^{\perp i}_j T^j_{\perp i} \\ & - 8T^{ji}_j T^\perp_{\perp i} - 2T_{j\perp}{}^i T^j_{\perp i} - 2T^i_{\perp j} T_{j\perp}{}^i + 4T^i_{\perp i} T^j_{\perp j}), \end{aligned} \quad (35)$$

here we have used the notation

$$V_\perp = n^\mu V_\mu,$$

and

$$V^\perp = n_\mu V^\mu.$$

Only the factors

$$T^I_{\perp i} = n^\mu T^I_{\mu i} = \frac{1}{N} (\dot{e}^I_i - \partial_i e^I_0 + N^j T^I_{ij}) \quad (36)$$

contain the time derivatives $\dot{e}^I_i = \mathcal{L}_t e^I_i$, the canonical momentum conjugate to e^I_i is

$$\tilde{p}_I{}^i = \frac{\partial \mathcal{L}_G}{\partial \dot{e}^I_i} = \frac{ec^4}{16\pi G} (T^{\perp i}_I - 2n_I T^{ji}_j - T_{I\perp}{}^i - T^i_{\perp I} + 2e_I{}^i T^j_{\perp j}), \quad (37)$$

which has the properties

$$\tilde{p}^{(ij)} = \tilde{p}_I{}^{(i} e^{I|j)} = \frac{ec^4}{8\pi G} (q^{ij} T^k_{\perp k} - T^{(i}_{\perp}{}^{j)}), \quad (38)$$

$$\tilde{p}^{[ij]} = \tilde{p}_I{}^{[i} e^{I|j]} = \frac{ec^4}{16\pi G} T^{\perp ij}, \quad (39)$$

and

$$\tilde{p}_{\perp}{}^i = n^I \tilde{p}_I{}^i = \frac{ec^4}{8\pi G} T^{ji}_j. \quad (40)$$

Using these results and

$$\tilde{p}_I^i \dot{e}^I_i = NT^I_{\perp i} \tilde{p}_I^i + N^j T^I_{ji} \tilde{p}_I^i + \tilde{p}_I^i \partial_i e^I_0,$$

$$\tilde{p}_I^i \partial_i e^I_0 = \partial_i (\tilde{p}_I^i e^I_0) - N n^I \partial_i \tilde{p}_I^i - N^j e^I_j \partial_i \tilde{p}_I^i,$$

$$NT^I_{\perp i} \tilde{p}_I^i = \frac{Nec^4}{16\pi G} (T^j_{\perp i} T^{\perp i}_j - 2T^{\perp}_{\perp i} T^{ji}_j - T^j_{\perp i} T_{j\perp}^i - T^j_{\perp i} T^i_{\perp j} + 2T^i_{\perp i} T^j_{\perp j}),$$

we can obtain the canonical Hamiltonian density

$$\begin{aligned} \mathcal{H}_G &= \tilde{p}_I^i \dot{e}^I_i - \mathcal{L}_G \\ &= N\mathcal{H}_{\perp} + N^j \mathcal{H}_j + \partial_i \tilde{B}^i, \end{aligned} \quad (41)$$

where

$$\begin{aligned} \mathcal{H}_{\perp} &= \frac{2\pi G}{ec^4} (\tilde{p}^2 - 2\tilde{p}^{(ij)} \tilde{p}_{(ij)}) - n^I \partial_i \tilde{p}_I^i \\ &\quad - \frac{ec^4}{64\pi G} (T_{ijk} T^{ijk} + 2T_{ikj} T^{jki} - 4T^i_{ki} T^{jk}_j - T^{\perp}_{ij} T^{\perp ij}) \\ &= \frac{2\pi G}{ec^4} (\tilde{p}^2 - 2\tilde{p}^{ij} \tilde{p}_{ji} - 2\tilde{p}_{\perp}^i \tilde{p}_{\perp i}) - \frac{ec^4}{64\pi G} (T_{ijk} T^{ijk} + 2T_{ikj} T^{jki}) - n^I \partial_i \tilde{p}_I^i, \end{aligned} \quad (42)$$

$$\mathcal{H}_j = T^I_{ji} \tilde{p}_I^i - e^I_j \partial_i \tilde{p}_I^i = \tilde{p}_I^i \partial_j e^I_i - \partial_i \tilde{p}_j^i, \quad (43)$$

$$\tilde{B}^i = \tilde{p}_I^i e^I_0 = t^{\mu} e^I_{\mu} \tilde{p}_I^i, \quad (44)$$

with $\tilde{p} = e^I_i \tilde{p}_I^i$. The term $\partial_i \tilde{B}^i$ in \mathcal{H}_G is essential for the quasilocal energy-momentum of the gravitational field [5] but it can be ignored in the constraint analysis.

The primary constraints are

$$\phi_N = \tilde{p}_N = \frac{\partial \mathcal{L}_G}{\partial \dot{N}} = 0, \quad (45)$$

$$\phi_i = \tilde{p}_i = \frac{\partial \mathcal{L}_G}{\partial \dot{N}^i} = 0, \quad (46)$$

and the primary Hamiltonian density is

$$\mathcal{H}_p = \mathcal{H}_G + \alpha \tilde{p}_N + \beta^i \tilde{p}_i. \quad (47)$$

For the further constraint analysis we need to compute

$$\begin{aligned} \frac{\delta \mathcal{H}_{\perp}}{\delta e^I_i} &= \frac{c^4}{16\pi G} \partial_j [e(T_I^{jj} + 2T^{[ij]}_I)] \\ &\quad + \frac{2\pi G}{ec^4} [e_I^i (\tilde{p}^2 - 2\tilde{p}^{jk} \tilde{p}_{kj} - 2\tilde{p}_{\perp}^j \tilde{p}_{\perp j}) + 2\tilde{p} \tilde{p}_I^i - 4\tilde{p}_I^j \tilde{p}_j^i - 4e_{Ij} \tilde{p}_{\perp}^j \tilde{p}_{\perp}^i] \\ &\quad - \frac{ec^4}{64\pi G} [e_I^i (T_{ijk} T^{ijk} + 2T_{ikj} T^{jki}) + 4(T_{jkI} T^{jki} + T_{jkI} T^{ikj} + T_{jkI} T^{kji})], \end{aligned} \quad (48)$$

$$\frac{\delta \mathcal{H}_{\perp}}{\delta \tilde{p}_I^i} = \frac{4\pi G}{ec^4} (\tilde{p} e^I_i - \tilde{p}^I_i - \tilde{p}_i^I) + \partial_i n^I, \quad (49)$$

$$\frac{\delta \mathcal{H}_j}{\delta e^I_i} = -\partial_j \tilde{p}_I^i, \quad (50)$$

and

$$\frac{\delta \mathcal{H}_j}{\delta \tilde{p}_I^i} = \partial_j e^I_i. \quad (51)$$

The consistency conditions

$$\dot{\phi}_N = \{\phi_N, H_p\} = -\frac{\delta H_p}{\delta N} = -\mathcal{H}_\perp = 0, \quad (52)$$

$$\dot{\phi}_N = \{\phi_i, H_p\} = -\frac{\delta H_p}{\delta N^i} = -\mathcal{H}_i = 0. \quad (53)$$

lead two secondary constraints, while the conditions

$$\dot{\mathcal{H}}_\perp = \{\mathcal{H}_\perp, H_p\} = 0,$$

$$\dot{\mathcal{H}}_i = \{\mathcal{H}_i, H_p\} = 0,$$

are only some conditions imposed on the Lagrange multipliers N, N^i, α, β^i and lead to no new constraints.

Thus by following the Dirac constraint analysis we find that the phase space $(\Gamma_{TG}, \Omega_{TG})$ of the teleparallel gravity is coordinatized by the pair (e^I_i, \tilde{p}_I^i) and has symplectic structure

$$\Omega_{TG} = \int_\Sigma d\tilde{p}_I^i \wedge de^I_i. \quad (54)$$

Ignoring the surface integral the Hamiltonian is given by

$$H_{TG} = \int_\Sigma N \mathcal{H}_\perp + N^j \mathcal{H}_j. \quad (55)$$

IV. CONSTRAINT ALGEBRA

In order to obtain the constraint algebra we construct the constraint functions by smearing \mathcal{H}_\perp and \mathcal{H}_i with test fields N and N^i on Σ following the approach of Ashtekar and Romano[8,17]:

$$C(N) = \int_\Sigma N \mathcal{H}_\perp, \quad (56)$$

$$C(\vec{N}) = \int_\Sigma N^i \mathcal{H}_i. \quad (57)$$

Under the gauge condition

$$\partial_i N^i = 0, \quad (58)$$

we have

$$\frac{\delta C(\vec{N})}{\delta e^I_i} = -\mathcal{L}_{\vec{N}} \tilde{p}_I^i, \quad (59)$$

and

$$\frac{\delta C(\vec{N})}{\delta \tilde{p}_I^i} = \mathcal{L}_{\vec{N}} e^I_i. \quad (60)$$

The Hamiltonian vector field of $C(\vec{N})$ generates a transformation with the parameter ε , the changes of e^I_i and \tilde{p}_I^i under this transformation are, respectively,

$$\delta e^I_i = -\{C(\vec{N}), e^I_i\}\varepsilon = \frac{\delta C(\vec{N})}{\delta \tilde{p}_I^i}\varepsilon = \varepsilon \mathcal{L}_{\vec{N}} e^I_i, \quad (61)$$

$$\delta \tilde{p}_I^i = -\{C(\vec{N}), \tilde{p}_I^i\}\varepsilon = -\frac{\delta C(\vec{N})}{\delta e^I_i}\varepsilon = \varepsilon \mathcal{L}_{\vec{N}}^I \tilde{p}_I^i, \quad (62)$$

which means that the Hamiltonian vector field of $C(\vec{N})$ on the phase space is the lift of the vector field N^i on Σ . One can say that the vector constraint $C(\vec{N})$ generates a space translation along the shift vector \vec{N} on Σ .

Using the Hamiltonian equations

$$\dot{e}^I_i = \frac{\delta H_p}{\delta \tilde{p}_I^i} = \frac{\delta C(N)}{\delta \tilde{p}_I^i} + \frac{\delta C(\vec{N})}{\delta \tilde{p}_I^i}, \quad (63)$$

and

$$\dot{\tilde{p}}_I^i = -\frac{\delta H_p}{\delta e^I_i} = -\frac{\delta C(N)}{\delta e^I_i} - \frac{\delta C(\vec{N})}{\delta e^I_i},$$

in the case $N^i = 0$, we have

$$\frac{\delta C(N)}{\delta e^I_i} = -\dot{\tilde{p}}_I^i = -\mathcal{L}_{\vec{t}} \tilde{p}_I^i, \quad (64)$$

and

$$\frac{\delta C(N)}{\delta \tilde{p}_I^i} = \dot{e}^I_i = \mathcal{L}_{\vec{t}} e^I_i. \quad (65)$$

The changes of e^I_i and \tilde{p}_I^i under the diffeomorphism with the parameter ε generated by the Hamiltonian vector field of $C(N)$ are, respectively,

$$\delta e^I_i = -\{C(N), e^I_i\}\varepsilon = \frac{\delta C(N)}{\delta \tilde{p}_I^i}\varepsilon = \varepsilon \mathcal{L}_{\vec{t}} e^I_i, \quad (66)$$

$$\delta \tilde{p}_I^i = -\{C(N), \tilde{p}_I^i\}\varepsilon = -\frac{\delta C(N)}{\delta e^I_i}\varepsilon = \varepsilon \mathcal{L}_{\vec{t}}^I \tilde{p}_I^i, \quad (67)$$

which means that when restricted to the constrained phase space the Hamiltonian vector field of $C(N)$ is the lift of the vector field $t^\mu = N n^\mu$ on Σ . In other words the scalar constraint $C(N)$ generates a lapse of time.

In order to calculate the Poisson brackets of $C(N)$ and $C(N)$, we derive two formulas. If $f(M)$ is any real-valued function on the phase space of the form

$$f(M) = \int_{\Sigma} M^{a\cdots b}{}_{c\cdots d} \tilde{f}_{a\cdots b}{}^{c\cdots d}(e^I_i, \tilde{p}_I^i), \quad (68)$$

where $M^{a\cdots b}{}_{c\cdots d} = M^{a\cdots b}{}_{c\cdots d}(\vec{x})$ is any tensor field independent of e^I_i and \tilde{p}_I^i on Σ then

$$\begin{aligned} \{C(\vec{N}), f(M)\} &= \int_{\Sigma} \frac{\delta C(\vec{N})}{\delta e^I_i} \frac{\delta f(M)}{\delta \tilde{p}_I^i} - \frac{\delta C(\vec{N})}{\delta \tilde{p}_I^i} \frac{\delta f(M)}{\delta e^I_i} \\ &= - \int_{\Sigma} \mathcal{L}_{\vec{N}} \tilde{p}_I^i \frac{\delta f(M)}{\delta \tilde{p}_I^i} + \mathcal{L}_{\vec{N}} e^I_i \frac{\delta f(M)}{\delta e^I_i} \\ &= - \int_{\Sigma} M^{a\cdots b}{}_{c\cdots d} \mathcal{L}_{\vec{N}} \tilde{f}_{a\cdots b}{}^{c\cdots d}. \end{aligned}$$

Integrating by parts and throwing away the surface integral, we obtain

$$\{C(\vec{N}), f(M)\} = f(\mathcal{L}_{\vec{N}} M). \quad (69)$$

Let $C(\vec{M})$, $C(M)$ to be $f(M)$ we have

$$\{C(\vec{N}), C(\vec{M})\} = C(\mathcal{L}_{\vec{N}}\vec{M}) = C([\vec{N}, \vec{M}]). \quad (70)$$

$$\{C(\vec{N}), C(M)\} = C(\mathcal{L}_{\vec{N}}M). \quad (71)$$

By the similar way we can get for $C(N)$:

$$\{C(N), f(M)\} = -\mathcal{L}_{\vec{t}}f(M) + f(\mathcal{L}_{\vec{t}}M) - f(\mathcal{L}_{\vec{N}}M). \quad (72)$$

Let $f(M) = C(M)$, we have

$$\{C(N), C(M)\} = C(\mathcal{L}_{\vec{t}}M) - C(\mathcal{L}_{\vec{N}}M), \quad (73)$$

owing to the consistency condition $C(M) = \mathcal{L}_{\vec{t}}C(M) = 0$.

The equations (70), (71) and (73) indicate that the constraint algebra is closed and the constraints $C(N)$ and $C(\vec{N})$ are first class, which is very similar to the case in general relativity. The equations (61), (62), (66) and (67) mean that the first class constraints $C(N)$ and $C(\vec{N})$ generate the corresponding gauge transformations, the spacetime translations. We have shown that under the gauge condition $\partial_i N^i = 0$ the constraint algebra of the teleparallel gravity has the same structure as that of general relativity.

V. CONCLUSION

From a common relation between the tetrad e^I_μ , the spin connection ω_μ^{IJ} , and the affine connection $\Gamma^\rho_{\nu\mu}$, it is proved that the gravitational energy-momentum in the teleparallel gravity can be expressed in terms of the Lorentz gauge potential as well as the translation gauge field strength. It is this characteristic that leads to the complicated transformation property of the gravitational energy-momentum: it is not covariant under local Lorentz transformations but is covariant under general coordinate transformations. The lack of a local Lorentz covariance can be considered as the teleparallel manifestation of the pseudotensor character of the gravitational energy-momentum in general relativity. It is not possible to define a local Lorentz covariant gauge current in the teleparallel gravity, consequently the corresponding gravitational energy-momentum in general relativity can not be represented by a true tensor.. Therefore the apparent covariance of the gravitational energy-momentum density is actually cosmetic. The quasilocal approach is the farthest we can go in the direction to deal with the problem of gravitational energy-momentum in the framework of gauge theories. The Hamiltonian formulation of the teleparallel gravity is the same as the Hamiltonian formulation of general relativity. Under the gauge condition $\partial_i N^i = 0$ the two constraints $C(N)$ and $C(\vec{N})$ are first class and generate the corresponding gauge transformations, the spacetime translations. The constraint algebra of the teleparallel gravity has the same structure as that of general relativity. Thus we have shown that the teleparallel gravity is equivalent to general relativity not only in the Lagrangian formulation but also in the Hamiltonian formulation although their geometries are different. In microscopic physics the teleparallel description is more useful because of its flat background structure.

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